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ON THE STABILITY OF STATIONARY MOTIONS OF SYSTEMS WITH QUASI-IGNORABLE COORDINATES AND OF MECHANICAL EQUILIBRIUM UNDER THE ACTION OF A MAGNETIC FIELD

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It has been shown that the stability of the steady-state motions of systems with quasi-ignorable coordinates can be judged from the stability of the equilibrium of a position subsystem with constant (invariable) quasi-ignorable generalized velocities. This allows us to disregard the degrees of freedom corresponding to the quasi-ignorable coordinates and to use Poincare's results on the change of stability at the bifurcations of equilibria by taking the quasi-ignorable velocities as parameters. Examples of systems of the class being considered are electromechanical systems not containing capacitances. The results mentioned above are valid for them also for a nonlinear connection between B and H in magnetics. In the case of ignorable coordinates the judgement on the stability of a stationary motion from the stability of the equilibrium of a position subsystem is possible with the aid of Routh's theorem generalized and supplemented by Liapunov [1]. However, this case differs essentially from the one being considered in that it is the ignorable momenta and not the velocities that are taken as constants.

1. Let there be given a system with stationary holonomic constraints, describable by m quasi-ignorable $(q_1, \ldots, q_m)$ and $n - m$ position $(q_{m+1}, \ldots, q_n)$ coordinates (according to [2], Chap. 7, Sect. 19, a coordinate is said to be quasi-ignorable if it does not occur in the expressions for the kinetic energy and for the generalized forces, while the generalized force corresponding to it is nonsero). We assume that two kinds of generalized forces correspond to quasi-ignorable coordinates: dissipative forces, depending only on the quasi-ignorable generalized velocities, and constant forces. We take the generalized forces corresponding to the position coordinates to be potential; we shall talk about the influence of dissipation on the position coordinates later on. A "position" subsystem can be a distributed-parameter system also. The system's kinetic potential L and the dissipation function F have the form

$$L = T - \Pi = T_1 + U + T_2 - \Pi$$

$$\Pi = \Pi(q_{m+1}, \ldots, q_n), \quad F = F(q_1^{\cdot}, \ldots, q_m^{\cdot})$$

$$T_1 = \frac{1}{2} \sum_{r, s=1}^{m} a_{rs}(q_{m+1}, \ldots, q_n)\, q_r^{\cdot} q_s^{\cdot}$$

$$U = \sum_{r=1}^{m} \sum_{s=1}^{n-m} a_{rm+s}(q_{m+1}, \ldots, q_n)\, q_r^{\cdot} q_{m+s}^{\cdot}$$

$$T_2 = \frac{1}{2} \sum_{r,\, s=1}^{n-m} a_{m+r\,m+s}(q_{m+1}, \ldots, q_n)\, q_{m+r}^{\cdot} q_{m+s}^{\cdot} \tag{1.1}$$

Two forms of motion are possible in the systems being examined

$$q_r^{\cdot} = h_r = \mathrm{const} \quad (r = 1, \ldots, m)$$
$$q_{m+r} = u_r = \mathrm{const} \quad (r = 1, \ldots, n-m) \tag{1.2}$$

where the constants h_r, u_r are determined from the equations

$$\frac{\partial F(h)}{\partial h_r} = e_r \qquad (r = 1, \ldots, m) \tag{1.3}$$
$$\frac{\partial \Pi(u)}{\partial u_r} = \frac{\partial T_1(h, u)}{\partial u_r} \qquad (r = 1, \ldots, n-m)$$

Here e_r are the constant generalized forces corresponding to the quasi-ignorable coordinates, $h = (h_1, \ldots, h_m)$, the designations $\Pi(u)$, $T_1(h, u)$ have an analogous meaning. For distributed-parameter systems the second group of equations in (1.3) should be undestood as the conditional notation of the equilibrium equations of the position subsystem under the action of the forces originating in the quasi-ignorable subsystem. According to (1.3) the quasi-ignorable velocities in the stationary motion do not depend on the position coordinates and they can be given arbitrary values (at least within certain limits) by varying the dissipation and the constant generalized forces. Therefore, when determining the possible positions of equilibrium of the position sybsystem it is permissible to reckon that the quasi-ignorable velocities are given directly. As a result we obtain the problem on the equilibrium of the position subsystem under the action of forces depending on the parameters; the quasi-ignorable subsystem is excluded from consideration. Let us show that the stability problem for the solutions (1.2) also reduces to the investigation of the stability of the equilibrium of the position subsystem under the assumption that the quasi-ignorable velocities are prescribed (nonvariable) parameters.

We introduce the perturbations η_r, ζ_r by the relations

$$q_r^{\cdot} = h_r + \eta_r^{\cdot}, \; r = 1, \ldots, m, \; q_{m+r} = u_r + \zeta_r, \; r = 1, \ldots, n-m,$$

and we write out the variational equations

$$\sum_{s=1}^{m} [a_{rs}(u)\eta_r^{\cdot\cdot} + b_{rs}(h)\eta_s^{\cdot}] + \sum_{s=1}^{n-m} [a_{r\,m+s}(u)\zeta_s^{\cdot\cdot} + g_{r\,m+s}(h, u)\zeta_s^{\cdot}] = 0 \quad (r = 1, \ldots, m$$

$$\sum_{s=1}^{n-m} [a_{m+r\,m+s}(u)\zeta_s^{\cdot\cdot} + (g_{m+r\,m+s}(h, u) - g_{m+s\,m+r}(h, u))\zeta_s^{\cdot} + c_{rs}(h, u)\zeta_s] + \tag{1.4}$$

$$\sum_{s=1}^{m} [a_{s\,m+r}(u)\eta_s^{\cdot\cdot} - g_{s\,m+r}(h, u)\eta_s^{\cdot}] = 0 \quad (r = 1, \ldots, n-m)$$

Here

$$g_{r\,m+s}(h, u) = \sum_{i=1}^{m} \frac{\partial a_{ri}(u)}{\partial u_s} h_i$$

$$g_{m+r\,m+s}(h, u) = \sum_{i=1}^{m} \frac{\partial a_{i\,m+r}(u)}{\partial u_s} h_i \tag{1.5}$$

$$b_{rs}=\frac{\partial^2F(h)}{\partial h_r\partial h_s},\quad c_{rs}=\frac{\partial^2\Pi(u)}{\partial u_r\partial u_s}-\frac{1}{2}\sum_{i,j=1}^{m}\frac{\partial^2a_{ij}(u)}{\partial u_r\partial u_s}h_ih_j$$

We investigate the stability with respect to the variables

$$q_1^{\cdot},\ldots,q_m^{\cdot},\ q_{m+1},\ldots,q_n$$

The original system is obtained from the conservative one after introducing in it dissipative forces with partial dissipation. Therefore, motion (1.2) is stable if the variational equations do not have unbounded solutions (undamped oscillations, i.e. pure imaginary roots, are admissible) and is unstable if unbounded solutions exist. We adopt the very same criterion also for distributed-parameter systems. We do not examine the case when time-independent solutions $\eta_1^{\cdot},\ldots,\eta_m^{\cdot},\zeta_1,\ldots,\zeta_{n-m}$ occur. We assume that dissipation with respect to the quasi-ignorable coordinates is total and that the matrix $\|b_{rs}\|$ is positive definite. Let us suppose that the equilibrium of the position subsystem, considered under the assumption that the quasi-ignorable velocities are constant, is unstable or possesses provisional stability. Let us suppose also that not one solution ζ_ν of the equations describing the small oscillations of the position subsystem with constant quasi-ignorable velocities

$$\sum_{s=1}^{n-m}[a_{m+rm+s}(u)\zeta_{\nu s}^{\cdot\cdot}+(g_{m+rm+s}(h,u)-g_{m+sm+r}(h,u))\zeta_{\nu s}^{\cdot}+c_{rs}(h,u)\zeta_{\nu s}]=0$$

$$(r=1,\ldots,n-m)\tag{1.6}$$

satisfies simultaneously the m conditions

$$\sum_{s=1}^{n-m}[a_{rm+s}(u)\zeta_{\nu s}^{\cdot\cdot}+g_{rm+s}(h,u)\zeta_{\nu s}^{\cdot}]=0\tag{1.7}$$

$$(r=1,\ldots,m)$$

Then the corresponding solution (1.2) is unstable.

Indeed, let us assume that it is stable. From the relation

$$dH_*/dt=-2F_*$$

$$H_*=T_1(u,\eta^{\cdot})+U(u,\eta^{\cdot},\zeta^{\cdot})+T_2(u,\zeta^{\cdot})+\Pi_*(u,\zeta)$$

$$F_*=\frac{1}{2}\sum_{r,s=1}^{m}b_{rs}\eta_r^{\cdot}\eta_s^{\cdot},\qquad \Pi_*=\frac{1}{2}\sum_{r,s=1}^{n-m}c_{rs}\zeta_r\zeta_s\tag{1.8}$$

it follows that in stable motion the $\eta_1^{\cdot}(t),\ldots,\eta_m^{\cdot}(t)$ are such that the integral

$$\int_{t_0}^{t}F_*(\tau)\,d\tau$$

is bounded as $t\to 0$. Let us show that in the given case system (1.4) cannot have only solutions satisfying the condition $\eta_r^{\cdot},\ \zeta_r\to 0$ as $t\to\infty$. Indeed, let us choose initial conditions such that we have $H_{*0}<0$. According to (1.8), $H_*\leqslant H_{*0}$ for all t. But if $\eta_r^{\cdot},\ \zeta_r\to 0$, then $H_*\to 0$, which contradicts the requirement $H_*\leqslant H_{*0}$. Thus, system (1.4) should have a particular solution in which $\zeta_s=z_s\cos(\lambda t+\psi_s)$ and at least a part of the z_s are nonzero, while the $\eta_s^{\cdot}$ are such that the integral of $F_*(t)$ is bounded as $t\to\infty$. Let us consider the second sum in one of the first m equations in (1.4). It is a linear form in $\zeta_s^{\cdot}$, $\zeta_s^{\cdot\cdot}$ and for the ζ_s of the indicated form either vanishes identically or turns into the function $Z_r \cos(\lambda t + \vartheta_r)$. The first sum in the equation being considered also has the very same form. But this sum is a linear form in $\eta_s^{\cdot}$, $\eta_s^{\cdot\cdot}$. Consequently, if both sums do not vanish identically, then at least one of the functions $\eta_s^{\cdot}$ should contain a term of the form $x_s \cos(\lambda t + \gamma_s)$. But for an $\eta_s^{\cdot}$ of such a form and for a positive-definite $\| b_{rs} \|$ the integral of F_* is not bounded. Therefore, in a stable motion all $2m$ sums mentioned should be identically equal to zero. By the same method we find that each of the $2(n-m)$ sums in the second group of equations in (1.4) should be identically equal to zero. As a result we have obtained that in the solution being considered the ζ_r satisfy both Eqs. (1.6) as well as conditions (1.7). But this is impossible by assumption. Consequently, system (1.4) cannot have only solutions where the integral of $F_*(t)$ is bounded, while the ζ_r are bounded or tend to zero as $t \to \infty$; this exhibits the instability.

Equalities (1.7) correspond to the cases when a part of the unknowns in (1.4) is found independently of the remaining ones. Here, as before, the instability of the position subsystem implies the instability of the whole system (1.4), but for provisional stability of the position subsystem the whole system can be both stable as well as unstable. Suppose that the position subsystem with constant quasi-ignorable velocities is unstable. If an unbounded solution ζ_ν satisfies conditions (1.7), then system (1.4) has an unbounded solution of the form $\eta^{\cdot} \equiv 0$, $\zeta = \zeta_\nu$, which signifies instability. Let us assume that a certain number of bounded solutions ζ_ν, ζ_μ, etc. satisfies conditions (1.7). Solutions of the form $\eta^{\cdot} \equiv 0$, $\zeta = \zeta_\nu, \zeta_\mu, \ldots$, are particular solutions of system (1.4). We consider the collection of its solutions which are linearly independent of these solutions. The preceding proof of instability is valid for them.

The following case is possible when the position subsystem is provisionally unstable. We examine the particular solutions ζ_ν satisfying the requirement that the inequality $T_2(\zeta_r^{\cdot}) + \Pi_*(\zeta_\nu) < 0$ be fulfilled for certain t. We assume that all such ζ_ν satisfy conditions (1.7). Then for any solution $\eta^{\cdot}$, ζ of system (1.4), linearly independent of all solutions of the form $\eta^{\cdot} \equiv 0$, $\zeta = \zeta_\nu$, the inequality $H_*(\eta^{\cdot}, \zeta, \zeta^{\cdot}) > 0$ is fulfilled for all t and the preceding proof of instability turns out to be invalid. In this (and only in this) case the provisional stability can be preserved for the whole system (1.4). A trivial example of the preservation of provisional stability is the case when all a_{sm+r}, g_{sm+r} equal zero and system (1.4) splits up into two unconnected subsystems. If a position subsystem with nonvariable quasi-ignorable velocities possesses secular stability, then solution (1.2) is stable.

Indeed, let us examine a system whose oscillations are described by equations of perturbed motion. The energy relation

$$dH_{*1}/dt = -2F_{*1} \tag{1.9}$$

$$H_{*1} = H_* + H_{*2}(\eta^{\cdot}, \zeta^{\cdot}, \zeta), \qquad F_{*1} = F_* + F_{*2}(\eta^{\cdot})$$

is valid for it, and the expansions of functions H_{*2}, F_{*2} in powers of their arguments start with terms of order higher than two. Therefore, for values of arguments sufficiently small in modulus, H_{*1} is positive definite, while its time derivative, taken by virtue of that equations of perturbed motion, is not positive.

If $g_{m+rm+s} - g_{m+sm+r} = 0$, $r, s = 1, \ldots, n-m$, for example, in the case of a gyroscopically unconnected system with $U = 0$, the equation of small oscillations of the position subsystem with constant quasi-ignorable velocities do not contain gyroscopic terms and provisional stability is impossible. In this case the stability of the stationary motions is uniquely determined by the equilibrium properties and does not depend on whether dissipation with respect to the position coordinates is taken into account. In the general case instability and secular stability are determined from the equilibrium properties (independently of dissipation in the position subsystem). But in this case the judgement on stability will be unique if the dissipative forces corresponding to the position coordinates are taken into account.

2. An important example of systems of the class being considered are electromechanical systems with closed conduction currents (i. e. not containing capacitances as well as slide contacts) In many cases such systems are sufficiently accurately described in a quasi-stationary approximation. If, further, we can take it that the dimensions of the cross sections of the conductors are small in comparison with their lengths and that the resistances do not depend upon the displacements, then the system has a kinetic potential and a dissipation function of the previous form. Here the role of quasi-ignorable coordinates is played by the charges, of the velocities, by the currents, the constant "quasi-ignorable" generalized forces are the external electromotive forces, and the dissipation in the quasi-ignorable coordinates is stipulated by the resistances of the conductors; the mechanical generalized coordinates are, however, position coordinates. The magnetic field energy corresponds to the term T_1 in the expression for the kinetic energy, while its derivatives with respect to q_{m+r} determine the ponderomotive forces. The expression for T usually does not contain the term U .

In the case of an electromechanical system the constant values of currents and mechanical equilibrium under the action of a constant magnetic field correspond to a stationary solution. In this case the results of Sect. 1 allow us to disregard the "electrical" degrees of freedom, in particular, the electrical circuit, the feed method, etc. are of no interest. They also permit us to judge the stability from the dependency of the form of equilibrium on the currents with the aid of Poincaré's theory of bifurcations.

The preceding discussion can be extended also to "magnetically nonlinear" electromechanical systems under the condition that hysteresis can be neglected. Here the expression for T differs from the one adopted earlier only in that a different form of the current function enters into it in the place of the quadratic form T_1. Therefore, it is sufficient to show that the form

$$\sum_{r, s=1}^{m} \frac{\partial T_1}{\partial h_r \partial h_s} \eta_r^{\cdot} \eta_s^{\cdot} \tag{2.1}$$

is positive definite. We restrict ourselves to the case when the induction vector $\mathbf{B}$ and the field strength vector $\mathbf{H}$ in magnetic theory are parallel, while the function $B\ (H)$ is an increasing function. For parallel $\mathbf{B}$ and $\mathbf{H}$

$$T_1 (h_1, \ldots, h_m) = \int dv \int_0^H B(H)\, dH \tag{2.2}$$

where the first integral is taken over the whole space. Let us find the increment $\Delta T_1 = T_1 (h + \eta^{\cdot}) - T_1 (h)$, by retaining the squares of $\eta_1^{\cdot}, \ldots, \eta_m^{\cdot}$. By $\Delta \mathbf{H} = \mathbf{H} (h + \eta^{\cdot}) - \mathbf{H} (h)$ we denote the increment in the field strength. Retaining the second-order terms in $\Delta \mathbf{H}$, we obtain

$$\Delta T_1 = \int dv \left[\mathbf{B}\Delta\mathbf{H} + \frac{\mathbf{H}^2(\Delta\mathbf{H})^2 - (\mathbf{H}\Delta\mathbf{H})^2}{2H^3} B + \frac{1}{2}\frac{dB}{dH}\left(\frac{\mathbf{H}\Delta\mathbf{H}}{H}\right)^2 \right] \qquad (2.3)$$

In the magnetically-nonlinear case $\Delta\mathbf{H}$ includes not only the first but also the higher degrees of increments of currents. But from the relation in [3] (Chap. IV, Sect. 32)

$$\int \mathbf{B}\Delta\mathbf{H} dv = \sum_{r=1}^{m} \Phi_r \eta_r^{\cdot} \qquad (2.4)$$

where Φ_r is the magnetic flux through the loop of the rth current, it follows that the integral of $\mathbf{B}\Delta\mathbf{H}$ is a linear form in the $\eta_1^{\cdot}, \ldots, \eta_m^{\cdot}$. Therefore, the value of form (2.1) is found from the last two terms in (2.3) if in them we replace $\Delta\mathbf{H}$ by the part of $\Delta\mathbf{H}$ which is linear in $\eta_1^{\cdot}, \ldots, \eta_m^{\cdot}$. This part vanishes in the whole space only for $\eta_1^{\cdot}, \ldots, \eta_m^{\cdot} = 0$. Taking into account the form of the last terms in (2.3) we conclude that form (2.1) is positive definite.

Under the assumption that the displacements do not change the resistances (a valid one, for example, if the conductors are fixed but the nonconductive magnetizable bodies move) it is possible to generalize also to the case when there are solid conductors. In this case we should use the expansions, mentioned in [4] (Chap. V), of the current density in terms of solenoidal functions of the space coordinates. Then the sole difference from the preceding discussion is that we obtain a system with a countable set of quasi-ignorable coordinates. In the case of ignorable coordinates judgement on the stability of the stationary motion from the stability of the equilibrium of the position subsystem can be made by means of Routh's theorem as generalized by Liapunov and of supplements to it [1]. In this case, however, it is the ignorable momenta and not the velocities which are taken as constants. The stability conditions obtained by means of the Routh theorem are broader than the stability conditions for the same system but with quasi-ignorable coordinates.

Let us show this for the case $U = 0$. By $p_1, \ldots, p_m$ we denote the ignorable momenta and by $V_R = \Pi + T_1(p, u)$ the Routh-altered force function. The solution $q_{m+r} = u_r =$ const, $r = n - m$, is stable if the quadratic form

$$\sum_{r,s=1}^{n-m} \frac{\partial^2 V_R}{\partial u_r \partial u_s} v_r v_s \qquad (2.5)$$

is positive definite, and is unstable if this form takes negative values for some $v_1, \ldots, v_{n-m}$. However, when the coordinates $q_1, \ldots, q_m$ are quasi-ignorable, we should insert into (2.5) the function $V = \Pi - T_1(h, u)$ instead of V_R. Suppose that the quantities $q_r^{\cdot} = h_r$, $r = 1, \ldots, m$, are the same in systems with ignorable and with quasi-ignorable coordinates. Then solutions exist where the u_r also are the same. For such solutions we set up the form (2.5) and the analogous form containing V, taking into account that the coefficient matrix $\| a_{rs}^{(-1)} \|$ in $T_1(p, u)$ is inverse to the matrix $\| a_{rs} \|$. As a result we obtain that the difference between (2.5) and the second form equals the nonnegative quantity

$$\sum_{r,s=1}^{m} a_{rs}^{(-1)} f_r f_s$$

where

$$f_r = \sum_{i=1}^{n-m} \sum_{s=1}^{m} \frac{\partial a_{rs}}{\partial u_i} h_s v_i$$

Among electromechanical systems Routh's theorem covers systems with superconductive loops. We disregard those exceptional cases when both the quadratic forms mentioned above vanish for one and the same $v_1, \ldots, v_{n-m}$. Then the preceding discussion signifies that the forms of equilibrium under the action of a magnetic field which are stable when the field is created by loops with finite conductivity, are stable also for superconductivity, but forms exist which are stable only in the case of superconductive loops. Systems with superconductive loops possess, consequently, qualitative singularities in the "purely mechanical" sense being considered here.

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A PRACTICAL METHOD FOR COMPUTING NORMAL FORMS IN NONLINEAR OSCILLATION PROBLEMS

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The applications of normal forms (see [1] for the history and a bibliography) to nonlinear oscillations have been outlined in [2]. As one of the applied problems we indicate the investigations of Ishlinskii ([3], Appendix 2) in [4]. One unsolved problem that remains is the derivation of recurrence formulas for computing the coefficients of the normalizing transformations and of the normal forms. These formulas have been derived below for a general case in the theory of oscillations (the absence of nonprime elementary divisors in the matrix of the linear part) on the basis of Briuno's theorem [1].

1. Statement of the problem. Let an oscillatory system be described by an nth-order autonomous system of differential equations, in which the variables can also be complex valued. We assume that the elementary divisors of the matrix of its linear part are prime. For oscillatory systems with Hermitian or unitary matrices of the linear part the latter condition is fulfilled by virtue of the Weierstrass theorem (for example, see [5], Sect. I.1.14). We shall assume that the original system has already been reduced to diagonal form and that its right-hand side is analytic in some neighborhood